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Probabilistic Component Mode Synthesis of Nondeterministic Substructures

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Standard methods of structural dynamic analysis assume that the structural characteristics are deterministic. Recognizing that these characteristics are actually statistical in nature, researchers have recently developed a variety of methods that use this information to determine probabilities of a desired response characteristic, such as natural frequency, without using expensive Monte Carlo simulations. One of the problems in these methods is correctly identifying the statistical properties of primitive variables such as geometry, stiffness, and mass. We present a method where the measured dynamic properties of substructures are used instead as the random variables. The residual flexibility method of component mode synthesis is combined with the probabilistic methods to determine the cumulative distribution function of the system eigenvalues. A simple cantilever beam test problem is presented that illustrates the theory.

Nomenclature

b	= boundary degrees of freedom
$[C]^a$	= correlation matrix of random variables in substructure a
g	= limit state function
i	= internal degrees of freedom
k	= number of kept modes per substructure
$[L]_c^a$	= Cholesky decomposition of $[C]^a$
N	= total number of degrees of freedom per substructure
p	= total number of substructures
p_f	= probability of failure
$p(x < X)$	= probability that x is less than X
$\{u\}$	= vector of uncorrelated standard normal random variables
X	= vector of random variables
X^*	= design or most probable point
$\{x\}$	= vector of correlated normal random variables
$\{x\}'$	= vector of correlated standard normal random variables
α	= substructure number

Introduction

STRUCTURAL analysts have always known that the parameters of the system being modeled are not deterministic, because of manufacturing tolerances, material deviation, and other factors. Until recently, the primary way to deal with this knowledge was to use safety factors, which are qualitative and based primarily on experience. In an effort to take account of these variations in the structural parameters in a more quantitative fashion, significant research has been performed to develop methods to actually use the statistical characteristics of the input quantities in the analysis to generate an output value that is also described statistically. Monte Carlo (MC) simulations can be performed to calculate these probability distributions, but up to a million runs are required for accurate results. Approximate techniques have therefore been developed that require several orders of magnitude fewer calculations than MC techniques. One such method, the fast probability integration (FPI) method,[1] has recently been implemented in a new probabilistic finite element code, NESSUS.[2]

Numerical analyses of structural vibration generally use the finite element method (FEM) as the basis for obtaining free and forced response characteristics. A frequent problem when using the FEM is that, for large models composed of many substructures, the number of degrees of freedom (DOFs) is so large that the computational costs are prohibitively expensive for eigenanalysis. This problem is particularly relevent when using probabilistic techniques, because the eigenanalysis has to be repeated many times. The favored solution method in industry for the deterministic situation is to apply dynamic component mode synthesis (CMS) methods, which substantially reduce the DOFs for the system model.

This paper defines a procedure for combining CMS with probabilistic methods to obtain the statistical characteristics in an efficient manner. These characteristic are summarized in the form of the cumulative distribution function (CDF). The procedure makes use of statistical distribution information of each substructure's dynamical modes and residual flexibility, which are available from experimental data. This information is synthesized into a system model using the residual flexibility method of CMS, and the statistics of the system dynamic characteristics are obtained using FPI. An advantage of this method over existing probabilistic structural analysis methods is that, in many cases, the statistics of the substructure dynamic characteristics may be easier to determine than those of primitive random variables (RVs) such as geometry, material stiffness, or density. In addition, these statistical dynamic characteristics completely incorporate all random factors in the structure. This is virtually impossible to do with primitive RVs such as thickness or density, since they are in reality random fields, varying over the entire structure. Final development of the method should allow probabilistic methods to be applied to much larger models than previously possible, such as turbomachinery bladed disks, which are composed of many almost identical substructures whose structural characteristics can be described statistically.

Probabilistic Theoretical Background

Research in the field of probabilistic structural mechanics has concentrated in two areas. The first can be described as perturbation

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methods, and the second as reliability methods. The perturbation method, as developed by Collins and Thomson, Kiefling, and Collins, Kennedy, and Hart, is used to derive an analytical expression for the mean and standard deviation of structural eigenvalues and eigenvectors as a function of the derivatives of the mass and stiffness matrix for each input RV.[3–5] Hasselman and Hart[6] used CMS to derive analytical expressions for the system, or global, eigenvalue matrix as a function of the modally reduced substructure stiffness and mass matrices. This approach is presently being studied by Mahadevan and Mehta.[7]

The research described in this paper employs the reliability method approach to determining the statistical structural response characteristics. To review this technique briefly, consider the scalar limit state function $g(\boldsymbol{X})$, defined as some value of interest that is a function of a vector of RVs. Cornell,[8] using the reliability approach, developed what would later be called the first-order reliability method (FORM) by truncating the higher-order terms (HOTs) from the series

$$g(\boldsymbol{X}) = g(\mu_X) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x_i - \mu_i) + \text{HOTs} \tag{1}$$

where μ_X is a vector of the mean values of the RVs in $\boldsymbol{X}$. This resulted in the following simple approximations for the mean value and standard deviation (σ) of g:

$$\mu_g \approx g(\mu_X) \tag{2}$$

$$\sigma_g \approx \sqrt{\sum \left(\frac{\partial g}{\partial x_i}\right)^2 \sigma_{x_i}^2} \tag{3}$$

Hasofer and Lind[9] refined and expanded this method further. They redefined g as

$$g(\boldsymbol{X}) = Y(\boldsymbol{X}) - y \tag{4}$$

where y is a specific value and the performance function $Y(\boldsymbol{X})$ is a function of the RVs. This formulation divides the space into two parts, $g < 0\,(Y < y)$ and $g > 0\,(Y > y)$. The probability that the function Y does not exceed the value y is the probability that $g < 0$. For example, if y equals an eigenvalue of interest, $p(g < 0)$ will be the probability that the actual eigenvalue obtained is less than the one of interest. There will therefore be a limit state function g for every y in the range of possibility. If the distributions of the RVs are normal and the resulting distribution of g is normal, then the following steps can be performed to obtain the probability that $g < 0\,(Y < y)$. First, a transformation of g to standard normal coordinates Z is made as follows:

$$Z = \frac{g(\boldsymbol{X}) - \mu_g}{\sigma_g} \tag{5}$$

and if β, which Cornell termed the safety index, is defined as

$$\beta = \mu_g/\sigma_g \tag{6}$$

then

$$p[g(\boldsymbol{X}) < 0] = p(Z < -\beta) = \Phi(-\beta) \tag{7}$$

where Φ is the CDF of the standard normal distribution function found in handbooks. Since negative values for β are not tabulated, the relationship

$$\Phi(-\beta) = 1 - \Phi(\beta) \tag{8}$$

is used instead to calculate this probability. The complete CDF is formed by finding the β values for all the limit states in the range of possibility. These values are only accurate, though, for linear limit states where the RVs have normal distributions. In addition, g could be formulated differently for some cases, thereby yielding different probabilities. To provide invariance with respect to the formulation

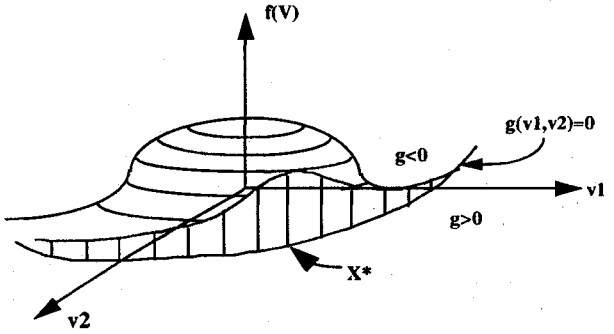

Fig. 1 Joint probability volume of two standard normal RVs.

of g, Hasover and Lind introduced an initial reduction of each of the primitive normal RVs x_i to standard normal RVs v_i using

$$v_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \tag{9}$$

This transformation allows the new limit states $g(\boldsymbol{V}) = 0$ to be plotted in standard normal joint probability space for every possible y. The joint probability of $g < 0$ will be the volume under the multidimensional bell-shaped surface over the area where $g < 0$. Furthermore, β can be shown to be the shortest distance from the origin to the line $g(\boldsymbol{V}) = 0$, and the point on the line at β can be called the most probable point (MPP), or design point $\boldsymbol{X}^*$, because that point will have the highest probability of occurrence of any point along the line $g(\boldsymbol{V}) = 0$ (see Fig. 1).

The reliability method was expanded by Rackwitz[10] to multidimensional, nonlinear limit states in which g is an explicit function of the RVs. Wu and Wirshing[11] developed the advanced mean value (AMV) method, a procedure for minimizing the number of iterations needed to obtain the MPP for nonlinear limit states; this is vital for nonexplicit limit states, such as finite element solutions. The limit state is approximated as a linear function about the means of the RVs [Eq. (1)], and the partial derivatives are approximately obtained by numerically differentiating the limit state with respect to each RV. Values of β and $\boldsymbol{X}^*$ are obtained for each desired limit state by using the FPI method, which is a compilation of the improvements to the FORM made by Wu,[11] Rackwitz,[10] and others. At this point in the procedure, an exact solution for each of the limit states (usually a finite element solution) is found by plugging in these most probable points. These results and their associated β values can then be used to create an entire CDF, which is shown in Wu's paper[11] to be in very good agreement with MC simulations for several examples. Further iterations can be performed by expanding the limit state about the new design points, instead of about the means as in Eq. (1). The FPI and AMV methods have been incorporated in NESSUS,[2] a probabilistic finite element program under development by NASA Lewis and the Southwest Research Institute.

Probabilistic Dynamic Synthesis

The proposed methodology makes use of the residual flexibility method of CMS. This method has been developed by MacNeil,[12] Craig and Chang,[13] and Martinez et al.[14] The essential idea in CMS is that substructure modes are truncated because their higher modes will not have a major effect on the system modes. The residual flexibility method incorporates the effects of the higher modes by determining their flexibility. A side benefit is that all the elements of the system stiffness matrix can be obtained from test and that the mass matrix can be closely approximated by a unity matrix in the nonboundary partition (equal in size to the number of kept modes, k). Since all the substructure information can be obtained from test, probabilistic data can be completely incorporated into the system matrices to obtain the system modes.

The first step of the probabilistic dynamic synthesis (PDS) method developed in this paper is to divide the model of a structure into substructures $\alpha = a, b, \ldots, p$. The physical displacement vectors of each substructure, which have either a subscript i denoting internal

DOFs or a subscript b denoting boundary DOFs, can be written as

$$\begin{Bmatrix} x_i \\ x_b \end{Bmatrix}^a, \begin{Bmatrix} x_i \\ x_b \end{Bmatrix}^b, \ldots, \begin{Bmatrix} x_i \\ x_b \end{Bmatrix}^p \tag{10}$$

where there are altogether $\dim x_i + \dim x_b = N$ DOFs for that substructure.

Each substructure is represented by n samples. Each sample is modally tested individually in a configuration such that the interface locations with other substructures are in a free condition. For substructure α, sample i, the test will yield eigenvalues λ_i^α and eigenvectors $\{\varphi_i^\alpha\}$. In addition, the boundary partition of the residual flexibility matrix $[G_{bb}]_i^\alpha$ is obtained from the measured boundary drive-point frequency response functions of the boundary coordinates.[15] For use in the PDS method, only the kept (nontruncated) eigenvalues, the boundary coordinates of the kept eigenvectors, and the boundary partition of the residual flexibility matrix are needed. These values can be combined into a single vector $\{x\}_i^\alpha$, defined as

$$\{x\}_i^\alpha = \begin{Bmatrix} \{\varphi_b\}^1 \\ \vdots \\ \{\varphi_b\}^k \\ \lambda^1 \\ \vdots \\ \lambda^k \\ \{G_{bb}\} \end{Bmatrix}_i^\alpha \tag{11}$$

where $\{\varphi_b\}^j$ is a vector of the boundary-node modal displacements for the jth mode, and λ^j is the jth eigenvalue of substructure α.

If the entire sample of substructure α is tested, $\{x\}^\alpha$ can therefore be defined as a vector composed of elements that are each α normally distributed RV with measured mean and standard deviation. Using Eq. (9), this vector is now converted to $\{x\}'^\alpha$, a vector of standard normally distributed RVs. In addition, there will be some degree of correlation between the RVs. These correlation values range from 0, or no correlation, to ± 1, or fully correlated, and can be easily calculated from the measured data. The values are placed in a correlation matrix $[C]^\alpha$ relating each element with every other element. For the probabilistic analysis, a set of independent standard normal RVs $\{u\}^\alpha$ will have to be obtained. This can be accomplished[16] by making an orthogonal transformation of $\{x\}'^\alpha$ with its Cholesky-decomposition lower triangular matrix $[L]_c^\alpha$ to uncouple the $\{x\}'$ coordinates, thereby creating $\{u\}^\alpha$. This can be expressed for substructures $\alpha = a, b, \ldots, p$ as

$$\{x\}'^\alpha = [L]_c^\alpha \{u\}^\alpha \tag{12}$$

The FPI algorithm requires that each independent RV be varied individually by some percentage of its standard deviation σ, which was chosen to be 50% for this development, while the other RVs are kept constant at their mean values. The choice of the size of the variation is somewhat arbitrary. Each of these cases is then back-transformed to form a corresponding case of the original correlated RVs. These are then plugged into the model to generate the limit state approximation [Eq. (1)] of the response value, which is used to obtain the β values and the design points X^*.

Since the distributions of the RVs are standard normal, 0.5σ will simply equal 0.5 for the RV to be varied. The first case is therefore

$$\{u\}^a = \begin{Bmatrix} 0.5 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \qquad \{u\}^m = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \qquad m = 2, \ldots, p \tag{13}$$

The next case will consist of the second element in $\{u\}^a$ equaling 0.5 and all the other elements of $\{u\}^a$ as well as all the elements of the other $\{u\}^m$ equaling zero, and so on.

For each case, the $\{u\}$ for each substructure is then transformed to the set of correlated standard normal RVs $\{x\}'$ using the transpose of $[L]_c^\alpha$ and then into the original RVs $\{x\}$ using Eq. (9). The new matrices $\{\lambda\}$, $[\Phi]$, and $[G_{bb}]$ are pulled out from $\{x\}$ and placed in substructure mass and stiffness matrices according to the residual flexibility formulation:

$$K^\alpha = \begin{bmatrix} \Lambda + \Phi_{bk}^T G_{bb}^{-1} \Phi_{bk} & -\Phi_{bk}^T G_{bb}^{-1} \\ \text{sym} & G_{bb}^{-1} \end{bmatrix}, \qquad M^\alpha = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{14}$$

where Λ is a diagonal matrix of the eigenvalues $\{\lambda\}$.

The system mass and stiffness matrices are now generated by directly coupling the substructure mass and stiffness matrices. This is accomplished by ordering the kept DOFs of each substructure sequentially in the system matrices and adding the boundary partitions together. The system eigenvalues are then obtained, and a single eigenvalue of interest is chosen. As each independent RV in the set of p vectors $\{u\}$ is varied, the numerical partial differentiation of the eigenvalue with respect to the RV can be calculated. These are used to generate the linear approximation of the limit state at the mean value [Eq. (1)], which is used by the FPI code to obtain the MPPs and a first mean-value solution for the CDF of the chosen system eigenvalue. The MPPs are plugged back into the substructures' mass and stiffness matrices, the system is resynthesized, and new, updated eigenvalue levels are obtained for each probability level, following the AMV method described by Wu and Wirshing.[11] These levels are then plotted to show the entire CDF.

Test Case

Analysis of a spring–mass system (Fig. 2) using the PDS method has been completed. The test system consists of two substructures, a and b, each having four DOFs. Five thousand samples of each substructure were created initially using standard MC techniques. To achieve complete probabilistic generality, each spring in the system was assigned a normal distribution with a mean of 200 and standard deviation of 10, and each mass was assigned a normal distribution with a mean of 1.0 and standard deviation of 0.5. Since this initial numerical simulation is performed to represent measurements of the dynamic characteristics of a physical population, any distribution can be chosen for the masses and springs as long as it is consistent for the full-up model MC case and the PDS case. Normal distributions were therefore chosen for convenience. The effect of nonnormal distributions will be discussed later. The MC random vectors were then used to create the mass and stiffness matrices for the substructures (5000 for each), and a modal analysis run on the substructure samples to obtain their eigenvalues $\{\lambda\}^\alpha$ and eigenvectors $[\Phi]^\alpha$. Three of the four modes for each substructure were kept for the analysis. The boundary partition of the $N \times N$ residual flexibility matrix $[G_{bb}]^\alpha$ was analytically calculated directly from the modes that had been chosen to be truncated, in this case just the highest one, using

$$[G] = \sum_{i=k+1}^{N} \frac{\{\varphi\}_i \{\varphi\}_i^T}{\lambda_i} \tag{15}$$

where, in this case, $k = 3$ and $N = 4$. The statistics on these dynamic characteristics and the correlation between them were then calculated. These statistics are listed in Table 1. The listed quantities comprise the vectors $\{x\}^a$ and $\{x\}^b$ as described in Eq. (11).

A distribution characterization routine[17] was also performed on the distributions to see if they could be characterized as normal, which is an assumption of the methodology outlined. Partial results

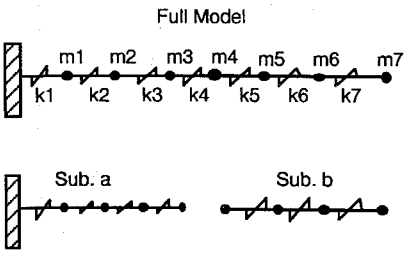

Fig. 2 Test-case system.

Table 1 Statistics of dynamic characteristics

Eigenvalues	Mean	Standard deviation
	Substructure A	
1	30.431	1.2702
2	246.74	10.336
3	552.89	23.014
	Eigenvectors, boundary location only	
1	0.7073	0.01155
2	−0.70725	0.0285446
3	0.70688	0.058827
	Residual flexibility (one boundary point only)	
1	6.46E−04	1.6159E−04
	Substructure B	
1	0.0	0.0
2	150.51	6.706
3	489.03	21.916
	Eigenvectors, boundary location only	
1	0.53466	6.9105E−03
2	0.75592	2.1945E−02
3	−0.55443	7.4042E−02
	Residual flexibility (one boundary point only)	
1	6.46E−04	1.5347E−04

Table 2 Partial results of distribution-types routine: substructure *B*, eigenvalue 2

W statistics (all types included)	
Normal	0.00948
Exponential	0.30339
Weibull	0.04448
EVD	0.04323
Lognormal	0.00934
Normal distribution parameters	
Sample mean = 150.61	
Sample std. dev. = 6.502	
Normal-distribution CDF fit to data	
Response value	CDF value
143.0	0.1210
147.0	0.2895
150.0	0.4628
155.0	0.7503
Lognormal parameters, base e	
$\mu = 5.014$	
$\sigma = 0.43201$	
Lognormal-distribution CDF fit to data	
Response value	CDF value
143.0	0.1193
147.0	0.2947
150.0	0.4713
155.0	0.7540

of the distribution characterization routine for one of the RVs are shown in Table 2. The W statistic is a goodness-of-fit test developed by Wirshing and Carlson, where a smaller number indicates a closer fit.[18] The results show that the data are well represented by both normal and lognormal distributions, with the lognormal being a little better. The CDF values for each distribution, however, indicate that the curves for the two distributions predict almost exactly the same value, which can be the case for a particular set of lognormal parameters. The assumption of normality was therefore deemed to be accurate.

At this point, the procedure follows the outline discussed previously for an actual case, which would use modal testing of physical samples to generate the dynamic characteristics and correlation matrix. For each substructure, a matrix composed of cases of $\{u\}$ vectors [see Eq. (14)] was generated and multiplied by the correlation Cholesky decomposition matrix $[L]_c$ to obtain $\{x\}'$, the set of correlated standard normal RVs. These were then converted to their nonstandard normal distributions and used in the residual flexibility substructure stiffness matrix. The substructures were then coupled

Table 3 Sample MPP output

Fund. eigenvalue response value = 8.101	
Probability = 0.01	
Most probable point	
Substructure-*a* RV	Value
u_1	−0.4863
u_2	−0.1692
u_3	−0.05685
u_4	−1.5807
u_5	−0.1221
u_6	−0.0269
u_7	0.211
Substructure-*b* RV	Value
u_1	−1.5205
u_2	0.2641
u_3	−0.2361
u_4	−0.0085
u_5	−0.3057
u_6	−0.0473
u_7	0.2267

Table 4 Dispersion error of PDS vs full MC model

CDF value	AMV	Δ from AMV mean	MC	Δ from MC mean	Error of Δ, %
0.010	8.116	−0.635	8.094	−0.613	−3.46
0.050	8.300	−0.451	8.266	−0.441	−2.23
0.100	8.399	−0.352	8.352	−0.355	0.87
0.159	8.476	−0.275	8.426	−0.281	2.22
0.200	8.520	−0.232	8.467	−0.240	3.62
0.500	8.751	0.000	8.707	0.000	Not applicable
0.800	8.985	0.234	8.944	0.237	1.44
0.841	9.029	0.278	8.987	0.280	0.74
0.900	9.108	0.357	9.072	0.365	2.25
0.950	9.210	0.459	9.169	0.462	0.65
0.990	9.403	0.652	9.344	0.637	−2.28

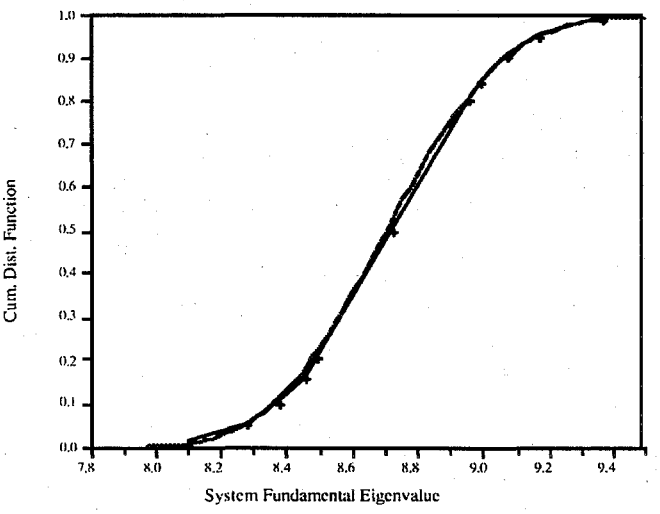

Fig. 3 CDFs for fundamental eigenvalue using PDS method and MC of full, unsubstructured model: –·+·–, pds amv and ——, full mod. mc.

together, and a modal analysis was performed on the system matrices. The first system eigenvalue for each case, which was the response value chosen, was then input along with its $\{u\}$ case into the FPI algorithm routine.

The output of the FPI routine is the mean-value solution, an initial estimate of the CDF of the response variable, and the MPPs for the specified CDF probability levels. One MPP from the output is shown in Table 3. Following the AMV procedure, these MPPs were recorrelated and converted to the original dynamic RVs as before, coupled, and a solution obtained for the updated fundamental eigenvalue. This value and its associated probability level were then used to create a new CDF.

For verification of the PDS method, an MC analysis was performed on the same nondeterministic spring–mass system, with the system eigenvalue directly obtained from each unsubstructured sample. The CDF for this full model is superimposed on the AMV CDF from the PDS method in Fig. 3. A very small amount of error is indicated graphically. To identify the error quantitatively, the amount of variation of the fundamental eigenvalue from its mean value at selected probability values for the PDS method was compared with the spread for the full model. The result, shown in Table 4, indicates that the deviations from the mean as computed by the AMV and MC methods agree to within 5%. In addition, the mean value of the fundamental eigenvalue computed by the AMV method is 8.751, which is only 0.5% higher than the one computed using MC (8.707), and the AMV standard deviation is 0.276, which is only 1.8% less than the MC standard deviation (0.281).

Concluding Remarks

A new methodology has been presented for performing analysis of structures composed of substructures whose dynamic characteristics can be identified statistically. This method uses the substructure eigenvalues, eigenvectors, and residual flexibility as random vectors for determining the desired response value by combining new probabilistic analysis techniques with the residual flexibility method of CMS. By this combination, improved computational efficiency is obtained as well as more confidence in the input statistics. Results for a test case show the method predicts close to the same CDF for the system fundamental eigenvalue as a nonsubstructured probabilistic MC analysis.

Future work on this method will include examining some basic conceptual questions on the limitations and applicability of the method. The fact that the dynamic RVs exhibited normal distributions may be a result of the assumption of normal distributions for the original primitive RVs. One extension of this research will be to examine the effect of nonnormal primitive RVs, which could result in nonnormal dynamic RVs. Equivalent normal distributions for these RVs would have to be obtained using methods developed by Chen and Lind and other researchers.[19] The methodology for handling the correlation of these nonnormal RVs will also have to be examined, as the correlation coefficient may not adequately describe the relationship between the variables. Since the research described in this paper is the first step in developing an entirely new methodology, the assumption of normal primitive RVs was necessary to avoid these complexities, and since normal distributions are commonly found, the results are applicable and worthwhile.

Another topic of interest is the formulation of a hybrid method combining analysis and test, since some of the dynamic characteristics necessary for the synthesis may be difficult to measure in some circumstances. Other questions include finding cases where the number of RVs can be reduced by perhaps only allowing stiffness or mass to vary, and examining the effect of boundary variability for situations like the fir-tree interface between blades and disks, which are neither fixed nor free. In addition, the method will be compared with perturbation methods to determine the areas of most efficient applicability for each one.

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